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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2018
FIRST YEAR [BATCH 2018-21]
MATHEMATICS (Honours)
Paper : I

Date : 14 /12/2018
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

Answer **any five** questions from Question Nos. 1 to 8 : [5×5]

1. Let ρ_1 and ρ_2 be two equivalence relations on a set A such that $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$. Prove that $\rho_1 \circ \rho_2$ is also an equivalence relation.

2. Let $f : A \to B$. Then show that f is left invertible if and only if f is injective.

3. a) Let G be a group and $a \in G$ such that $o(a) = n$. Show that if $a^m = e$ for some $m \in \mathbb{N}$ then n divides m.
   b) Let $G = \{a \in \mathbb{R} : -1 < a < 1\}$. Define a binary operation '$*$' on G by $a * b = \dfrac{a+b}{1+ab}$ for all $a, b \in G$. Show that $(G, *)$ is a group. (2+3)

4. a) If $\beta \in S_7$ and $\beta^4 = (2\ 1\ 4\ 3\ 5\ 6\ 7)$ then find β.
   b) Find the order of the permutation $(1\ 9\ 6\ 4)(2\ 5) \in S_{10}$. (4+1)

5. If H and K are subgroups of a group G then show that HK is a subgroup of G if and only if $HK = KH$.

6. a) Find all elements of order 8 in the group $(Z_{24}, +)$.
   b) Give an example of a group $(G, \circ)$ in which o(a), o(b) are finite but $o(a \circ b)$ is infinite for some $a, b \in G$. (3+2)

7. a) Show that every group of prime order is cyclic.
   b) Let G be a group such that $|G| < 319$. Suppose G has subgroups of order 35 and 45. Find the order of G. (2+3)

8. a) Prove that a group of order 27 must have a subgroup of order 3.
   b) Let A and B be two subgroups of a group G. If $|A| = p$, a prime integer then show that either $A \cap B = \{e\}$ or $A \subseteq B$. (3+2)

Answer **any five** questions from Question Nos 9 to 16 : [5×5]

9. a) Find the closure of the set of all rationals in $\mathbb{R}$.
   b) Prove that for any set $A \subseteq \mathbb{R}$, $\bar{A}^c = \left(A^o\right)^c$ where A^c denotes the complement of A in $\mathbb{R}$. (2+3)

10. a) Let A be a non-empty set of real numbers bounded below and if the set ($-A$) is defined by $-A=\{-x : x \in A\}$ then show that $Inf\ A = -Sup\ (-A)$.

b) If $a, b \in \mathbb{R}$ and $0 \le a-b < \varepsilon$ holds for every positive ε then prove that $a = b$. (3+2)

11. Evaluate the following limits:

a) $\lim\limits_{n\to\infty} \dfrac{2^{n+1}}{n!}$

b) $\lim\limits_{x\to\frac{1}{2}} f(x)$ where (2+3)

$$f(x)=\begin{cases} x \quad , \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$$

12. a) If $\{x_n\}$ and $\{y_n\}$ are two bounded sequences then prove that $\overline{\lim}(x_n+y_n) \le \overline{\lim}\ x_n + \overline{\lim}\ y_n$.

b) Examine whether the sequence $\left\{1+(-1)^n\right\}_n$ is convergent or not. (3+2)

13. a) Show that the limit, $\lim\limits_{x\to 0} \cos\dfrac{1}{x^2}$ does not exist.

b) Prove that $\lim\limits_{x\to 0}(1+x)^{1/x} = e$. (2+3)

14. Let $\{x_n\}$ be a sequence of real numbers where $x_1 = 6$ and for any $n \in \mathbb{N}$, $x_{n+1} = \sqrt{6x_n}$. Does $\lim\limits_{n\to\infty} x_n$ exist? Justify your answer.

15. a) Let $A=\left\{\dfrac{1}{2^m}+\dfrac{1}{3^n} \middle| m, n \in \mathbb{N}\right\}$. Find the closure of A.

b) Prove that a countable subset of $\mathbb{R}$ has an empty interior. (4+1)

16. a) Show by Cauchy's general principle of convergence that the sequence $\left\{\dfrac{n-1}{n+1}\right\}_n$ is convergent.

b) Show that the set of all integers is countable. (3+2)

Group – B

Answer **any three** questions from Question Nos. 17 to 21 : [3×5]

17. If the lines $ax^2+2hxy+by^2 = 0$ be the two sides of a parallelogram and the line $lx + my = 1$ be one of its diagonals, show that the equation of the other diagonal is $(am-hl)x = (bl-hm)y$.

18. Determine the values of a and f so that the equation $ax^2-2xy+y^2-6x+2fy+9=0$ may represent a conic having infinitely many centres and then determine its type. (2+3)

19. Show that the equation of the circle which passes through the focus of the parabola $\frac{2a}{r}=1+\cos\theta$ and touches it at the point $\theta=\alpha$ is given by $r\cos^3\left(\frac{\alpha}{2}\right)=a\cos\left(\theta-\frac{3\alpha}{2}\right)$.

20. If the sum of the ordinates of two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is b, show that the locus of the pole of the chord which joins them is $b^2x^2+a^2y^2=2a^2by$.

21. Let OP and OQ be two semi-conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then prove that

i) $OP^2+OQ^2=a^2+b^2$ and ii) the area of the parallelogram with OP and OQ as adjacent sides is ab. (3+2)

Answer **<u>any two</u>** questions from <u>Question Nos. 22 to 24</u> : [2×5]

22. a) Find the moment of a force represented by $\hat{i}+3\hat{j}$ acting through the point $3\hat{i}+2\hat{j}-\hat{k}$ about the point $-\hat{i}+\hat{j}+2\hat{k}$.

b) Let PQRS be a tetrahedron with $(-5,-4,8)$, $(2,3,1)$, $(4,1,2)$, $(6,3,7)$ as the co-ordinates of P, Q, R, S respectively. Find the distance of the point P from the plane through Q, R, S. (2+3)

23. a) Decompose a vector $\vec{r}$ as a linear combination of a vector $\vec{a}$ and another vector perpendicular to $\vec{a}$ and coplanar with $\vec{r}$ and $\vec{a}$.

b) Find the shortest distance between the lines $\vec{r}=\vec{r}_1+t\vec{\alpha}$, $\vec{r}=\vec{r}_2+t\vec{\beta}$ where $\vec{r}_1=(1,-2,3), \vec{\alpha}=(2,1,1), \vec{r}_2=(-2,2,-1)$ and $\vec{\beta}=(-3,1,2)$. (2+3)

24. a) Find the equation of the plane through the point (2, 2, 3) and parallel to the vectors $\hat{i}-2\hat{j}+4\hat{k}$ and $3\hat{i}+2\hat{j}-5\hat{k}$.

b) Solve the vector equation for $\vec{r}$ if $t\vec{r}+\vec{r}\times\vec{a}=\vec{b}$ where t is a non-zero given scalar and $\vec{a},\vec{b}$ are two given vectors. (2+3)

Answer **<u>any five</u>** questions from <u>Question Nos. 25 to 32</u> : [5×5]

25. Find an integrating factor of the differential equation $(y^2+2x^2y)dx+(2x^3-xy)dy=0$ and hence solve it.

26. Reduce $x^2p^2+y(2x+y)p+y^2=0$ to Clairaut's form by the substitution y = u, $xy=v$ where $p=\frac{dy}{dx}$. Hence solve the equation and prove that y + 4x = 0 is a singular solution.

27. Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ where λ is the parameter and a, b are constants.

28. a) Show that the general solution of the equation $\frac{dy}{dx}+Py=Q$ can be written in the form $y=K\left(y_1-y_2\right)+y_2$, where K is a constant, y_1 and y_2 are its two particular solutions.

b) Prove or disprove: The functions e^x, cos x, sin x are linearly independent. (3+2)

29. Solve the following differential equation by the method of undetermined coefficients:

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=x^3+\sin x .$$

30. Solve the following differential equation by the method of variation of parameters:

$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=\log x(x>0)$, it being given that y=x and $y=\frac{1}{x}$ are two linearly independent solutions of the associated homogeneous differential equation.

31. By the use of symbolic operator D find the solution of $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$.

32. Show that $\sin x\frac{d^2y}{dx^2}-\cos x\frac{dy}{dx}+2y\sin x=0$ is exact and solve it completely. (1+4)

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